AN INTEGRAL EQUATION ENCOUNTERED IN THE PROBLEM OF A RIGID FOUNDATION BEARING ON NONHOMOGENEOUS SOIL

(OB ODNOM INTEGRAL'NOM URAVNENII, VSTRECHAIUSHCHEMSIA V ZADACHE O DAVLENII ZHESTKOGO FUNDAMENTA NA NEODNORODNYI GRUNT)

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It was shown in an article by Klein [1] that if the deformation modulus of the soil varies with depth z according to the law $E = E_m z^m$ (where E_m is a constant), then provided that the index m and the coefficient of lateral expansion ν satisfy the relation $\nu(2 + m) = 1$, there exists an elementary solution which satisfies the St. Venant compatibility conditions and expresses the effect of a concentrated force applied in a direction normal to the surface of the soil. From this we can find a power kernel for the integral representing the settlement w in terms of the pressure $p = f(x, y)$

$$w = \vartheta F(x, y) = \vartheta \iint_D \frac{f(x', y')\, dx'dy'}{[(x - x')^2 + (y - y')^2]^{1/2(1+m)}} \qquad (0.1)$$

Here D is the area of contact

$$\vartheta = \alpha / \pi E_m, \qquad \alpha = \frac{1}{2}(3 + m) / (1 + m)(2 + m)$$

For non-negative values of m, integral (0.1) converges if, and only if, $0 \leqslant m < 1$. In cases when the above relation between the constants m and ν is not satisfied Formula (0.1) ceases to be exact, but can still be accepted as a basis for practical computations. In such cases the constant a is found from the values of the independent variables m, ν with the aid of a set of curves [1]. Although the results are not exact, Equation (0.1) and the unknown function $p = f(x, y)$ are of considerable interest in themselves, and in one or other equivalent form have already been the subject of investigation [2,3]. In particular, a solution is given in [3] in closed form for a circular area. We shall give below a solution for an elliptical area for the case when $F(x, y)$ is a polynomial, together with a solution for a circular area derived by a different method, by reducing (0.1) to an equation of the Abel type. As an example we shall consider a bearing surface in the form of a paraboloid of revolution.

1. In the case of an elliptical area the solution is based on the following theorem.

Theorem. Let the pressure under the bearing surface, which is elliptical in plan, be expressed by the product of the polynomial $\Pi(x, y)$ and the function

$$\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2(m-1)}$$

i.e.

$$p = f(x, y) = \Pi(x, y)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2(m-1)} \tag{1.1}$$

The settlement w is then given by a polynomial of the same order as $\Pi(x, y)$.

This theorem is a generalization of the analogous theorem of Shtaerman [4] which refers to the classical case of $m = 0$. It enables us to apply the method of indeterminate coefficients.

Proof. We introduce a set of polar coordinates with pole at a point (x, y) inside the ellipse, i.e. we set

$$x' = x + \rho\cos\varphi, \qquad y' = y + \rho\sin\varphi \tag{1.2}$$

We then have

$$F(x, y) = \int_0^{2\pi} d\varphi \int_0^{\rho_1(\varphi)} \Pi(x'y')\left(1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right)^{1/2(m-1)} \rho^{-m}\, d\rho \tag{1.3}$$

where $x'y'$ under the integral sign must be replaced by Expressions (1.2), and the function $\rho_1(\phi)$ in the upper limit is the positive root of the equation

$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} - 1 = \frac{(x + \rho\cos\varphi)^2}{a^2} + \frac{(y + \rho\sin\varphi)^2}{b^2} - 1 = 0 \tag{1.4}$$

which, for convenience, we shall write in the form

$$C\rho^2 + 2B\rho - A = 0 \tag{1.5}$$

where

$$A = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \geqslant 0, \qquad B = B(\varphi) = \frac{x \cos\varphi}{a^2} + \frac{y \sin\varphi}{b^2}$$
$$C = C(\varphi) = \frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2} \tag{1.6}$$

The roots of this equation are

$$\rho_1 = \rho_1(\varphi) = \frac{1}{C}(-B + \sqrt{B^2 + AC}) = \alpha \qquad (\alpha > 0)$$
$$\rho_2 = \rho_2(\varphi) = \frac{1}{C}(-B - \sqrt{B^2 + AC}) = -\beta \qquad (\beta > 0) \tag{1.7}$$

and, in addition

$$1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2} = C(\rho_1 - \rho)(\rho - \rho_2)$$

Further, we note that if we add π to the argument of ϕ we do not alter the value of the function $C(\phi)$, but the function $B(\phi)$ changes sign. Consequently

$$\rho_1(\varphi + \pi) = -\rho_2 = \beta, \qquad \rho_2(\varphi + \pi) = -\rho_1 = -\alpha$$

If we now divide the interval of integration with respect to ϕ into two parts, from 0 to π and from π to 2π, and replace ϕ by $\pi + \psi$ in the second part, we obtain on the basis of the foregoing remarks

$$F(x, y) = \int_0^\pi C^{1/2(m-1)} d\varphi \left\{ \int_0^\alpha \frac{\Pi(x + \rho\cos\varphi, y + \rho\sin\varphi)}{[(\alpha - \rho)(\beta + \rho)]^{1/2(1-m)}} \rho^{-m} d\rho + \right.$$
$$\left. + \int_0^\beta \frac{\Pi(x - \rho\cos\varphi, y - \rho\sin\varphi)}{[(\alpha + \rho)(\beta - \rho)]^{1/2(1-m)}} \rho^{-m} d\rho \right\} \tag{1.8}$$

In the classical case of $m = 0$ it is not difficult to combine the integrals in braces to form an integral of an analytic function (see [4]). In cases when $m \neq 0$, however, we proceed as follows. Consider the integral

$$J(\alpha, \beta) = \oint \Pi(x + \zeta\cos\varphi, y + \zeta\sin\varphi)\, [(\zeta - \alpha)(\zeta + \beta)]^{1/2(m-1)} \frac{d\zeta}{\zeta^m} \tag{1.9}$$

taken in the positive direction over the boundary containing the points α, $-\beta$, 0 in the ζ-plane cut along the real axis from α to $-\beta$.

If we contract this boundary towards the cut until it coincides with the edges of the discontinuity, to which should be added circles of arbitrarily small radius with centers in the points α, $-\beta$, 0, and if we then evaluate by the usual method the resulting integrals, we find that $J(\alpha, \beta)$ differs from the sum of the integrals in braces in (1.8) by a factor $2i \cos 1/2\, m\pi$. Consequently

$$F(x, y) = \frac{1}{2i \cos {}^1/_2 m\pi} \int_0^\pi C^{1/2(m-1)} J(\alpha, \beta)\, d\varphi \tag{1.10}$$

If we now expand $\Pi(x + \zeta \cos \phi,\ y + \zeta \sin \phi)$ in powers of ζ we find that

$$J(\alpha, \beta) = \sum_{n=0}^{N} \frac{1}{n!}\left(\cos\varphi \frac{\partial}{\partial x} + \sin\varphi \frac{\partial}{\partial y}\right)^n \Pi(x, y) \oint \zeta^{n-m} [(\zeta - \alpha)(\zeta + \beta)]^{1/2(m-1)} d\zeta \tag{1.11}$$

Here N is the order of the polynomial $\Pi(x, y)$ in the unknowns x, y. We shall set

$$\Phi_n(\alpha, \beta) = \frac{1}{2\pi i} \oint \zeta^{n-m} [(\zeta - \alpha)(\zeta + \beta)]^{1/2(m-1)} d\zeta \tag{1.12}$$

Then

$$F(x, y) = \frac{\pi}{\cos \frac{1}{2} m\pi} \int_0^\pi \sum_{n=0}^{N} \frac{\Phi_n(\alpha, \beta)}{n!} \left(\cos\varphi \frac{\partial}{\partial x} + \sin\varphi \frac{\partial}{\partial y}\right)^n \Pi(x, y) \frac{d\varphi}{C^{1/2(1-m)}} \tag{1.13}$$

Integral (1.12) is a finite expression. Indeed, $|\zeta| > \max(\alpha, \beta)$, and in order to evaluate this integral we expand the integrand into a series in powers of ζ^{-1}. But

$$\zeta^{n-m} [(\zeta - \alpha)(\zeta + \beta)]^{1/2(m-1)} = \zeta^{n-1} [(1 - a\zeta^{-1})(1 + \beta\zeta^{-1})]^{1/2(m-1)} =$$
$$= \sum_{r=0}^{\infty} \sum_{s=0}^{\infty} \frac{(-1)^r}{r!s!} \left(\frac{m-1}{2}\right)_r \left(\frac{m-1}{2}\right)_s \alpha^r \beta^s \zeta^{n-(r+s)-1} \tag{1.14}$$

All terms in which $r + s \neq n$ vanish after integration. Therefore

$$\Phi_n(\alpha, \beta) = \sum_{r=0}^{n} \frac{(-1)^r}{r!(n-r)!} \left(\frac{m-1}{2}\right)_r \left(\frac{m-1}{2}\right)_{n-r} \alpha^r \beta^{n-r} \tag{1.15}$$

If we now combine terms equidistant from the beginning and end of this series and take into account Expressions (1.7) for a and β, we find that, for n even or odd, in both cases terms containing square roots disappear. $\Phi_n(a, \beta)$ is therefore a polynomial in (x, y). Furthermore, we have found that the order of this polynomial in x, y is n. Integral (1.9) is therefore a polynomial of order N, as it was required to prove. We could have given at this stage the final expressions for $\Phi_n(a, \beta)$ in terms of x, y, but they are extremely cumbersome. For small values of n the expressions are

$$\Phi_0(\alpha, \beta) = 1, \qquad \Phi_1(\alpha, \beta) = (m-1)\frac{B(\varphi)}{C(\varphi)} \tag{1.16}$$

$$\Phi_2(\alpha, \beta) = \frac{1}{2}(m-1)(m-3)\left(\frac{B(\varphi)}{C(\varphi)}\right)^2 - \frac{1}{2}(m-1)\left(\frac{A}{C(\varphi)}\right) \quad (1.17)$$

2. Let us consider a flat bearing surface [die] inclined to the surface of the soil; here $w = w_0 + wx$. In accordance with the previous example we shall take $p = f(x, y)$ in the form

$$p = (p_0 + \mu x)\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)^{1/2(m-1)} \quad (2.1)$$

Applying the formulas of the previous example, we have

$$w = \frac{\pi\vartheta}{\cos\frac{1}{2}m\pi}\left\{p_0\int_0^\pi\left(\frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2}\right)^{1/2(m-1)} d\varphi + \right. \quad (2.2)$$

$$\left.\mu x\left[\int_0^\pi\left(\frac{\cos^2\varphi}{a^2} - \frac{\sin^2\varphi}{b^2}\right)^{1/2(m-1)} d\varphi + \frac{m-1}{a^2}\int_0^\pi \cos^2\varphi\left(\frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2}\right)^{1/2(m-3)} d\varphi\right]\right\}$$

From this p_0 and μ can easily be found in terms of w_0 and w. For $a \neq b$ an actual calculation would require tables of functions analogous to the total elliptical integrals, in the sense that the square root is replaced by $1/2(1 - m)$.

3. We shall give now the solution for a circular area. We shall use polar coordinates with pole at the point (0, 0) and expand $F(x, y)$ and $f(x, y)$ in Fourier series in the angular coordinate ϕ. It can easily be shown that the relation between the Fourier coefficients $F_n(r)$ and $f_n(r)$ in both expansions is

$$F_n(r) = \int_0^a s f_n(s)\, ds \int_0^{2\pi} \frac{\cos n\omega\, d\omega}{(r^2 + s^2 - 2rs\cos\omega)^{1/2(m+1)}} \quad (3.1)$$

Here we can write either $F_n(r)$ or $f_n(r)$ for $\cos n\phi$ and $\sin n\phi$, since in the present context this does not lead to any ambiguity.

The kernel of this equation

$$\chi_n(r, s) = \int_0^{2\pi} \frac{\cos n\omega d\omega}{(r^2 + s^2 - 2rs\cos\omega)^{1/2(m+1)}} \quad (3.2)$$

can easily be expressed in terms of an integral which gives a canonical representation of some hypergeometric function $F(a, b, c; z)$. For, suppose $r > s$, and let us put $h = s/r$ and $\zeta = e^{i\omega}$. Then

$$\chi_n(r, s) = \frac{-i}{r^{m+1}} \oint \frac{\zeta^{1/2(m+1)+n-1} d\zeta}{(1 - h\zeta)^{1/2(m+1)} (\zeta - h)^{1/2(m+1)}} \quad (3.3)$$

where the integral is taken over the circle $|\zeta| = 1$. If we now pass by the usual method from this integral to an integral taken along the part of the real axis $0 \leqslant x \leqslant h$ which forms the discontinuity, we find that

$$\chi_n(r, s) = \frac{2\cos {}^1/_2\, m\pi}{r^{m+1}} \int^{h} \frac{x^{1/2(m+1)+n-1} dx}{(1-hx)^{1/2(m+1)}(h-x)^{1/2(m+1)}} \tag{3.4}$$

It is convenient here to make the substitution $x = (h/s^2)t^2$. This gives

$$\chi_n(r, s) = \frac{4\cos {}^1/_2\, m\pi}{r^n s^n} \int_0^s \frac{t^{m+2n} dt}{[(r^2-t^2)(s^2-t^2)]^{1/2(m+1)}} \quad \text{for} \quad s \leqslant r \tag{3.5}$$

Also, by symmetry

$$\chi_n(r, s) = \frac{4\cos {}^1/_2\, m\pi}{r^n s^n} \int_0^r \frac{t^{m+2n} dt}{[(r^2-t^2)(s^2-t^2)]^{1/2(m+1)}} \quad \text{for} \quad r \leqslant s \tag{3.5a}$$

Substituting these expressions in Equation (1.2) we obtain

$$F_n(r) = \frac{4\cos {}^1/_2 m\pi}{r^n} \left\{ \int_0^r s^{1-n} f_n(s)\, ds \int_0^s \frac{t^{m+2n} dt}{[(r^2-t^2)(s^2-t^2)]^{1/2(m+1)}} + \right.$$
$$\left. + \int_r^a s^{1-n} f_n(s)\, ds \int_0^r \frac{t^{m+2n} dt}{[(r^2-t^2)(s^2-t^2)]^{1/2(m+1)}} \right\} \tag{3.6}$$

Since the singularity of the integrands (in the inner integrals) is non-essential, we can reverse the order of integration (in the first term by using Dirichlet's formula). Then, after adding the results, we obtain

$$F_n(r) = \frac{4\cos {}^1/_2 m\pi}{r^n} \int_0^r \frac{t^{m+2n} dt}{(r^2-t^2)^{1/2(m+1)}} \int_t^a \frac{s^{1-n} f_n(s)\, ds}{(s^2-t^2)^{1/2(m+1)}} \tag{3.7}$$

It will now be seen that in order to find $f_n(s)$ we simply have to solve two equations of the Abel type. As a final result we have

$$f_n(r) = \frac{-\cos {}^1/_2 m\pi}{\pi^2} r^{n-1} \frac{d}{dr} \int_r^a \frac{u^{1-m-2n} du}{(u^2-r^2)^{1/2(1-m)}} \frac{d}{du} \int^u \frac{s^{n-1} F_n(s)\, ds}{(u^2-s^2)^{1/2(1-m)}} \tag{3.8}$$

This formula coincides with the solution given in [3]. It can be simplified by integrating by parts and taking the differentiation under the integral sign (assuming that the derivative $F_n'(r)$ is continuous over the interval $0 \leqslant r \leqslant a$). Then

$$f_n(r) = \frac{\cos \frac{1}{2} m\pi}{\pi^2} r^n \left\{ \frac{\psi_n(a)}{(a^2 - r^2)^{\frac{1}{2}(1-m)}} - \int_r^a \frac{\psi_n'(u)\,du}{(u^2 - r^2)^{\frac{1}{2}(1-m)}} \right\} \qquad (3.9)$$

where

$$\psi_n(u) = u^{1-m-2n} \int_0^u \frac{[s^n F_n(s)]'\,ds}{(u^2 - s^2)^{\frac{1}{2}(1-m)}} \quad (n \geqslant 1) \qquad (3.10)$$

$$\psi_0(u) = F_0(0) + u^{1-m} \int_0^u \frac{F_0'(s)\,ds}{(u^2 - s^2)^{\frac{1}{2}(1-m)}} \quad (n = 0) \qquad (3.11)$$

We note also that for the settlement outside the bearing surface Formula (3.7) gives

$$F_n(r) = \frac{4 \cos \frac{1}{2} m\pi}{r^n} \int_0^a \frac{t^{m+2n}\,dt}{(r^2 - t^2)^{\frac{1}{2}(m+1)}} \int_t^a \frac{s^{1-n} f_n(s)\,ds}{(s^2 - t^2)^{\frac{1}{2}(m+1)}} \qquad (3.12)$$

By putting w in the form of a polynomial we see that (3.9) can be evaluated without difficulty.

4. We shall consider now a bearing surface in the form of a paraboloid of revolution

$$w = w_0 - Ar^k \qquad (k \geqslant 1) \qquad (4.1)$$

Here the index k is not necessarily an integer. The case of $k = 1$ corresponds to a conical [die] bearing surface. From Formula (3.11) we obtain

$$\psi_0(u) = \vartheta^{-1} \left(w_0 - \frac{\Gamma(\frac{1}{2} + \frac{1}{2}m)\,\Gamma(1 + \frac{1}{2}k)}{\Gamma(\frac{1}{2}m + \frac{1}{2}k + \frac{1}{2})} Au^k \right) \qquad (4.2)$$

Thus, a regular solution is possible on condition that

$$w_0 = \frac{\Gamma(\frac{1}{2} + \frac{1}{2}m)\,\Gamma(1 + \frac{1}{2}k)}{\Gamma(\frac{1}{2}m + \frac{1}{2}k + \frac{1}{2})} Aa^k \qquad (4.3)$$

This solution is of the form

$$p = K \int_\rho^1 \frac{t^{k-1}\,dt}{(t^2 - \rho^2)^{\frac{1}{2}(1-m)}}, \qquad \rho = \frac{r}{a} \qquad (4.4)$$

where the constant

$$K = \frac{\cos \frac{1}{2} m\pi}{\pi^2 \vartheta} \frac{\Gamma(\frac{1}{2} + \frac{1}{2}m)\,\Gamma(1 + \frac{1}{2}k)}{\Gamma(\frac{1}{2}m + \frac{1}{2}k + \frac{1}{2})} Aa^{m+k-1} \qquad (4.5)$$

We then have for the load P

$$P = \frac{2\pi K a^2}{(1+m)(k+1+m)} \tag{4.6}$$

$$p(r) = (1+m)(k+1+m)\frac{P}{2\pi a^2}\int_{\rho}^{1} \frac{t^{k-1}\,dt}{(t^2-\rho^2)^{1/2(1-m)}} \tag{4.7}$$

In the case when k is even this result can be expressed very simply. For example, when $k = 2$ (the analogue of the Hertz case) we find that

$$p(r) = P\frac{3+m}{2\pi a^2}(a^2-r^2)^{1/2(1+m)} \tag{4.8}$$

The distribution of pressure conforms very closely to this result in the classical case. With a conical bearing surface, however, the pressure $p(0)$ at the center remains finite. In this case the curve of the pressure has a vertical tangent at the center.

From (3.12) we can show that the settlement outside the bearing surface is

$$w_0(r) = \frac{2\cos\frac{1}{2}m\pi}{\pi}\,\frac{\Gamma(\frac{1}{2}+\frac{1}{2}m)\,\Gamma(1+\frac{1}{2}k)}{\Gamma(\frac{1}{2}m+\frac{1}{2}k+\frac{1}{2})}\,A\int_0^a \frac{t^m(a^k-t^k)}{(r^2-t^2)^{1/2(m+1)}}\,dt \tag{4.9}$$

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